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Immune-Motivated Optimization

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An immune-motivated optimization method is proposed and applied to both spin glass and graph coloring problems. Also an immune-motivated modification for the numerical Euler method is proposed.

KEY WORDS: immune; optimization; forward random Euler's method.

1. METADYNAMICS OF THE IMMUNE SYSTEM

The immune system (IS) is an interesting dynamical system (Farmer *et al.*, 1986). Every period, a certain fraction of its cells is eliminated and new cells are recruited. Typically, every day approximately 5% of its cells are replaced by new cells. This can be modeled by a dynamical process where the parameters change on a slow scale while the states change on a faster one. Such systems are called metadynamical systems (MDS).

Definition (Ahmed and Abdusalam, 1994). An MDS is defined by

$$z_{k+1} = f_i(z_k),$$

where $(i-1)N \leq k \leq iN$, $i = 1, 2, 3, \ldots$, and N is a large positive number that is chosen as follows: Fix $\varepsilon > 0$, then if the sequence $\{z_k\}$, $(i-1)N \leq k \leq iN$, is in the basin of an attractor $z_i^\star$ of f_i, then there exists $N' < iN$, such that $k > N'$ implies that $|z_k - z_i^\star| < \varepsilon$.

Intuitively speaking, this means that for all i, the transient system comes very close to its asymptotic value. For the logistic map $f_i(z_k) = \lambda_i z_k(1 - z_k)$, numerical

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simulations have shown that $N = 50$ is an acceptable value. Motivated by the IS, we are interested in the following updating of the parameters

$$\lambda_{i+1} = \omega\lambda_i + (1 - \omega) \times \text{rnd} \times \lambda_{\max}, \tag{1}$$

where both rnd, $\omega \in (0, 1)$, rnd is a uniformly distributed random number, and $\lambda_{\max}$ is the maximum allowed value for the parameter λ. The typical values of the IS correspond to $\omega \simeq 0.95$, and $\lambda_{\max} = 4$ for the logistic map. Starting by $\lambda_i \in (1, 3)$ and using the above values, one gets that for almost all $i > 1, \lambda_i \in (1, 3)$. Thus for most of the time, the system oscillates between the attractor $1 - 1/\lambda_i$. This oscillatory behavior has been observed experimentally in the immune system (Lundkvist *et al.*, 1989). Moreover this dynamics is used to explain the long-term immune memory even if the antigen (bacteria or virus) has been totally eradicated (Ahmed and Abdusalam, 1994; Head, 2000).

In the following section, we see that this idea of MDS improves the convergence properties of some numerical dynamical systems.

2. NEWTON'S FORMULATION

Newton's method for solving the nonlinear equation $g(x) = 0$ is given by the iteration

$$x_{k+1} = x_k - g(x_k)/g'(x_k), \quad k = 1, 2, 3, \ldots \tag{2}$$

This method has a fast local q-quadratic rate of convergence under some reasonable assumptions (Dennis and Schnabel, 1983). However, if the initial value x_1 is not close enough to the actual root, then the system (2) may not converge or even worse, it may converge to a spurious solution.

The dynamical system (2) has been studied for $g(x)$ polynomial in Hurley and Martin (1984) where it has been shown that if $g(x)$ has three or more distinct real roots then there are regions at which the system is chaotic. A similar study has been done for complex polynomials (Saupe, 1988).

This weakness has been solved by, at least, two methods (Dennis and Schnabel, 1983). One of them is reviewed here. Its similarity with immune metadynamics is interesting. The algorithm is to minimize the function $f(\text{r})$, where r is a vector of variables. This can be related to (2) by choosing $f = \frac{1}{2}g^2$.

Algorithm 1. *Fix* $a \in (0, .5)$ (*typically* $\alpha = 10^{-4}$) *and* $\rho \in (0, 1)$ *and choose* r_1 *arbitrary. Set* $k = 1$.

1. *Set* $\lambda_k = 1$.
2. *Compute* $r_{k+1} = r_k - \lambda_k H_k^{-1} \nabla f(r_k)$, *where* $H_k = [\frac{\partial^2 f(x_k)}{\partial x_i \partial x_j}] + \mu_k I$, *and* μ_k *is chosen such that* H_k *is safely positive definite.*

3. *If* $f(r_{k+1}) - f(r_k) \leq -\alpha\lambda_k \nabla f(r_k)^T H_k^{-1} \nabla f(r_k)$ *then set* $k = k + 1$ *and go to step* (1). *Else set* $\lambda_k = \rho\lambda_k$ *and go to step* (2).

Under the standard assumptions of Newton's method, this algorithm is globally convergent. The theorems proving the global convergence of this algorithm can be found in Dennis and Schnabel (1983).

3. CHAOS CONTROL

Chaos systems exhibit extreme sensitivity to initial conditions (Holmgren, 1996). Hence, chaos control is an important subject. Several attempts have been made (Ahmed *et al.*, 1999; Ott *et al.*, 1990). Motivated by the above algorithm, the following procedure for chaos control is proposed: Assume that $x_{k+1} = g(x_k, \alpha)$ is a chaotic dynamical system and that it is required to stabilize the system around a given value, say $x_\star$. This is equivalent to minimizing $f = \frac{1}{2}(g - x_\star)^2$. Using Algorithm 1, the parameter α should be perturbed according to the rule

$$\alpha_{k+1} = \alpha_k - \left[\lambda_k (g_k - x_\star)\frac{\partial g_k}{\partial a_k}\right] \Bigg/ \left[\left(\frac{\partial g_k}{\partial a_k}\right)^2 + (g_k - x_\star)\frac{\partial^2 g_k}{\partial a_k^2} + \mu_k\right], \qquad (3)$$

where $g_k = g(x_k, \alpha_k)$, and μ_k is chosen such that the denominator is safely positive. Notice that λ_k can be replaced by $\rho\lambda_k$ according to Step 3 in Algorithm 1. The advantage of this method is that the global convergence implies that one does not have to wait till the system gets close to $x_\star$, which was the case of OGY (Ott *et al.*, 1990).

The same procedure can be extended to find periodic orbits of period n via minimizing $f = \frac{1}{2}(g^n(x, \alpha) - x)^2$, using (4). The global convergence of (3) guarantees that it is more stable than the one used in Miller and Yorke (2000).

4. FORWARD RANDOM EULER'S METHOD

Relation (2) is used in forward Euler's method to solve the system $\frac{dp}{dx} = -xp$, $p(0) = 1$ numerically. We call this modification forward random Euler's (FRE) method.

It is known (Holmgren, 1996) that the standard forward Euler's method fails for this problem. The reason is that Euler's method gives the discrete system $p_{k+1} = (1 - \lambda^2 k)p_k$. Thus for all λ, there exists N such that $k > N$ implies that $|(1 - \lambda^2 k)| > 1$. When FRE method is applied, we obtain the correct asymptotic solution $p_k \to 0$ as k is increased.

5. IMMUNE MOTIVATED OPTIMIZATION

Nature has been created in a fascinating way. As we learn more, we found that it is more efficient to imitate it. Recently, this approach has been applied

to optimization (Boettcher and Percus, 2001). The authors proposed extremal optimization (EO) where fitness are assigned to different configurations. Then the one with the lowest fitness is removed and placed with another one randomly. This idea is motivated by the work of Bak and Sneppen (1993). It has been applied to the spin glass problem which consists of a d-dimensional hypercubic lattice of length L where at each site there is a spin $x_i \in \{-1, 1\}$, $i = 1, 2, \ldots, L^d$. The Hamiltonian of the system is given by

$$H = -\frac{1}{2}\Sigma_{\langle i,j\rangle} B_{ij} x_i x_j, \tag{4}$$

where the uniform distributed random variable $B_{ij} \in \{-1, 1\}$ and $\langle i, j\rangle$ means that j is the nearest neighbor (nn) to i. The objective is to choose the set $\{x_i\}$ that minimizes the energy H. The approach of EO is to assign fitness f_i to each site as follows:

$$f_i = \frac{x_i}{2}\Sigma_{j(\text{nn of } i)} B_{ij} x_j.$$

Here $H = -\Sigma_i f_i$. The EO algorithm is as follows:

Algorithm 2. *Choose* $\{x_i\}$ *randomly to form an initial configuration S. Repeat the following steps as desired.*

1. *Find* f_i *for all* x_i.
2. *Find the site* $i_\star$ *with lowest fitness and choose another configuration* S' *randomly such that* $x_{i_\star}$ *is replaced by another state.*
3. *If* $H(S') < H(S)$ *then* $S_{best} = S'$, *else* $S_{best} = S$.

The weakness of this approach is that focusing only on the worst fitness can lead to a narrow deterministic process. To overcome this weakness, the authors replaced Step 2 of the above algorithm by ranking the sites in an ascending order according to their fitness. Then the replaced site is chosen randomly according to the probability distribution.

$$p_k \propto k^{-\tau}.$$

The first proposed immune motivated optimization IMOP1 replaces Step 2 by

2. *Replace randomly the sites with lowest* 5% *fitness (not just the one with lowest fitness).*

This gives a significantly higher fitness than the previous deterministic one (at the same time) but it still has the previous drawback of falling into local minima.

Table I. The Average Cost/Vertex for Different Values of n and p

	$np = 3.6$	$np = 4$	$np = 4.5$	$np = 5$	$np = 5.5$	$np = 6$
$n = 32$	0.313	0.313	0.313	0.469	0.531	0.750
$n = 64$	0.313	0.422	0.516	0.609	0.906	0.906
$n = 96$	0.5	0.442	0.516	0.609	0.906	0.1063

Another immune motivated optimization algorithm IMOP2 which preserves the advantage of the above modification is to define α_i by

$$\alpha_i = \left[\frac{\bar{\lambda} - \lambda_i}{\bar{\lambda} - \underline{\lambda}}\right]^{\tau}, \tag{5}$$

where $\underline{\lambda}(\bar{\lambda})$ is the minimum (maximum) possible fitness and $\tau > 0$. Step 2 in the above algorithm is now replaced by

2. *If RND* $< \alpha_i$ *then replace* x_i,

where RND is a uniformly distributed random number. This algorithm has the following advantages: (1) It has a better chance of avoiding getting stuck in local minima. (2) It does not require the ranking of all fitness at each time step as the modification of Boettcher and Percus requires. This saves significant time specially for large number of sites.

Comparing IMOP1 against IMOP2 for the three-dimensional spin glass problem with $L = 10$, we obtained energy/site $\simeq -1.164$ for IMOP1 and $\simeq -1.352$ for IMOP2 after short time. After longer time, the value for IMOP2 has improved to $\simeq -1.70$ while that for IMOP1 did not improve. This was expected since IMOP1 is deterministic.

Now IMOP is applied to the problem of random graph coloring. It is studied for three colors but the generalization to any number of colors is direct. A random graph is generated by connecting any pair of its n vertices by an edge with probability p. Define the local cost at each vertex c_i to be (1/2) (number of its monochromatic nearest neighbors). Then if rnd $< [c_i/(np)]^{\tau}$, ($\tau = .3$ gave the best results in our simulations) then the site's color is changed randomly. The total cost of the configuration is defined by $C = \sum_i c_i$. If the total cost of the new configuration is less than the old one then the new configuration is preserved and the old one is discarded and vice versa. This process is continued for the time allowed. Our results for the average cost/vertex for 5000 runs are shown in Table I.

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